PMM U.S.S.R., Vol. 46, No. 6, pp. 819-824, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 539.383

THREE-DIMENSIONAL CONTACT PROBLEM OF THE MOTION OF A STAMP WITH FRICTION*

L.A. GALIN and I.G. GORIACHEVA

There is considered the three-dimensional contact problem of elasticity theory with friction forces collinear to the motion direction. Such a case holds during stamp motion along the boundary of an elastic half-space with anisotropic friction /1/. In the case of an arbitrary friction surface, the mentioned force distribution is satisfied approximately during stamp motion.

Taking account of the friction force is essential in such problems since it permits determination of the moment acting on the moving stamp because of the nonsymmetric pressure distribution. Known methods of solving three-dimensional contact problems with friction forces are, as a rule, for the axisymmetric case /2,3/, while certain solutions are obtained under the assumption of an axisymmetric pressure distribution on the contact area with a nonsymmetric friction distribution /4,5/. Recently, numerical methods of solving a similar species of problems started to be developed /6/.

The displacement is considered of a rigid stamp along the boundary of a rigid half-space in the direction of the x axis. We shall consider the problem quasistatic, which imposes definite constraints on the velocity v of stamp displacement, and we introduce a coordinate system (x, y, z) coupled to the moving stamp. We shall consider the friction forces on the contact area to be directed opposite to the stamp motion (along the ox axis). The stamp is here displaced so that it cannot turn under the effect of the applied forces. The boundary conditions will have the form

$$\begin{gathered} w = g(x, y) + \delta, \quad \tau_{xz} = \mu\sigma_z, \ \tau_{yz} = 0, \quad x, y \in \Omega \\ \sigma_z = \tau_{xz} = \tau_{yz} = 0, \quad x, y \overline{\in} \Omega \end{gathered} \tag{1}$$

Here μ is the friction coefficient, $g(x, y)$ is the shape of the stamp, δ is its seat, and Ω is the contact domain.

Stamp motion in the direction of the z axis can be represented as the superposition of displacements of points of the base caused by the application of a normal pressure $p(x, y)$ and of displacements due to the action of a tangential force. It follows from the solution of the problem of an elastic half-space subjected to lumped forces with components along the oz axis (T_z) and the ox axis (T_x) applied at the origin that vertical displacements of points of the boundary half-plane $(z = 0)$ are determined by the formula

$$w = \frac{(1-\nu^2)}{\pi E}\frac{T_z}{R} + \frac{(1+\nu)(1-2\nu)}{2\pi E}\frac{xT_x}{R^2}, \quad R = \sqrt{x^2+y^2} \tag{2}$$

Integrating (2) over the whole contact area Ω and taking account of conditions (1), we obtain the following integral equation to determine the pressure $p(x, y)$ under the stamp:

$$\begin{gathered} \iint_\Omega p(\xi, \eta)\left[\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} + \mu\alpha\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}\right] d\xi\, d\eta = \\ \frac{\pi E}{1-\nu^2}[g(x, y) + \delta], \quad \alpha = \frac{1-2\nu}{2-2\nu} \end{gathered} \tag{3}$$

The coefficient α will be zero when $\nu = 0.5$, i.e., the elastic body is incompressible. In this case, the presence of friction forces will not affect the magnitude of the normal pressure. For real bodies the Poisson's ratio ν takes a value $0 < \nu < 0.5$, hence the quantity varies between the limits 0.5 and 0, where $\alpha = 0.286$ for the Poisson's ratio $\nu = 0.3$. Moreover, it should be taken into account that the magnitude of the friction coefficient μ is also small. For dry friction of steel on steel $\mu = 0.2$. Considering $\nu = 0.3$, in this case $\mu\alpha \simeq 0.057$. For oiled surfaces the quantity $\mu\alpha$ takes a still smaller value.

Let $\mu\alpha = \varepsilon$ and let us consider this quantity as a certain small parameter. When the pressure on the contact area is bounded everywhere, the solution of the integral equation (3) for $\mu\alpha = 0$ can be taken as the zero-th approximation $p_0(x, y)$. We shall examine solutions $p(x, y)$ of (3) close to $p_0(x, y)$ when the parameter ε is small. In this case we represent

*Prikl.Matem.Mekhan.,Vol.46,No.6,pp.1016-1022,1982 the required function $p(x, y)$ in the form of the series

$$p(x, y) = p_0(x, y) + \varepsilon p_1(x, y) + \ldots + \varepsilon^n p_n(x, y) + \ldots \tag{4}$$

Substituting the series (4) into the fundamental integral equation (3), we obtain a recurrent system of equations to determine the unknown functions $p_n(x, y)$. Introducing the operator notation

$$A(\omega) = \iint_\Omega \omega(\xi, \eta) \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \quad B(\omega) = -\iint_\Omega \omega(\xi, \eta) \frac{(x-\xi)\, d\xi\, d\eta}{(x-\xi)^2 + (y-\eta)^2}$$

we write this system in the form

$$A[p_n(\xi, \eta)] = B[p_{n-1}(\xi, \eta)], \quad n = 1, 2 \ldots \tag{5}$$

To prove the convergence of the series (4) we construct its majorant number series. For definiteness we consider the contact area to have the shape of a circle of radius a (Ω is $x^2 + y^2 \leqslant a^2$).

The operator A is a linear bounded operator that conformally maps the space of functions continuous in the domain Ω into itself one-to-one. For any $x, y \in \Omega$ the norm of the operator A is uniformly bounded

$$\| A \| = \sup_{\|\omega\| \leqslant 1} \| A(\omega) \| = \sup_{r \leqslant a} 4 \int_0^{\pi/2} \sqrt{a^2 - r^2 \sin^2 \varphi}\, d\varphi = 2a\pi$$

The mutual one-to-oneness of the mapping follows from the fact that the equation

$$\int_0^{2\pi} \int_0^a \rho\omega(\rho, \varphi) L(r, \rho, \theta, \varphi)\, d\rho\, d\varphi = 0$$

$$L(r, \rho, \theta, \varphi) = [r^2 + \rho^2 - 2r\rho \cos(\varphi - \theta)]^{-1/2}$$

has only a trivial solution because of the symmetry of the kernel $L(r, \rho, \theta, \varphi)$ and the completeness of its eigenvalue system in the space $L_2(\Omega)$ /7,8/.

Therefore, all the conditions are satisfied for the Banach theorem on the inverse operator /9/, from which it follows that the operator A has a bounded inverse operator A^{-1}. The form of this operator is presented in /2/. If f is a smooth function, and ω is a bounded smooth function, then the operator A^{-1} has the form

$$\omega = A^{-1}(f) = \frac{1}{2\pi^3} \iint_\Omega \frac{\Delta f(\xi, \eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \times \operatorname{arctg} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2}}{a\sqrt{(x-\xi)^2 + (y-\eta)^2}}\, d\xi\, d\eta \tag{6}$$

The norm of the operator B is also bounded and the following estimate is valid

$$\| B \| = \sup_{\|\omega\| \leqslant 1} \| B(\omega) \| = \sup_{\Omega: x^2 + y^2 \leqslant a^2} \iint_\Omega \frac{|x - \xi|\, d\xi\, d\eta}{(x-\xi)^2 + (y-\eta)^2} =$$

$$\sup_{\substack{r \leqslant a \\ 0 \leqslant \theta \leqslant 2\pi}} 2 \int_0^\pi |\cos\varphi| \sqrt{a^2 - r^2 \sin^2(\varphi - \theta)}\, d\varphi = 4a$$

$$(x - \xi = \rho\cos\varphi, \ y - \eta = \rho\sin\varphi)$$

We multiply each equation of the system (5) by the inverse operator A^{-1}. Then

$$p_n(x, y) = A^{-1}B[p_{n-1}(x, y)] = C[p_{n-1}(x, y)] \tag{7}$$

The operator C is also bounded, and the following inequality is valid (k is a certain number)

$$\| C \| \leqslant \| A^{-1} \| \, \| B \| = k$$

Therefore, and estimate follows from (7) for terms of the series (4)

$$\| p_n(x, y) \| \leqslant \| C \| \, \| p_{n-1}(x, y) \| = k \, \| p_{n-1}(x, y) \|$$

that indicates that the series (4) converges uniformly for all $\varepsilon < 1/k$.

The dimensions of the contact area (the radius of the circle a) are determined from the condition $p(x, y) = 0$ on the boundary of the contact area Ω.

As an illustration, let us consider the problem about the motion of a smooth axisymmetric stamp of circular planform $g(r) = Dr^2$ on the boundary of an elastic half-space. We consider the contact area to be a circle of radius a. As is known /2/, in this case the function $p_0(x, y) = p_0(r)$ is determined from the formula

$$p_0(r) = \frac{4ED}{\pi(1 - \nu^2)} \sqrt{a^2 - r^2}$$

To find the next term in the series (4), we obtain

$$B[p_0(r)] = -\int_0^{2\pi}\int_0^a \frac{4ED}{\pi(1-\nu^2)} \frac{\sqrt{a^2-\rho^2}\,(r\cos\theta-\rho\cos\varphi)\,\rho\,d\rho\,d\varphi}{r^2+\rho^2-2r\rho\cos(\theta-\varphi)} =$$

$$b(r)\cos\theta, \qquad b(r) = \frac{8ED}{3(1-\nu^2)r}\left[(a^2-r^2)^{3/2}-a^3\right]$$

As follows from /10/, the solution of the equation

$$A[p_1(r,\theta)] = b(r)\cos\theta \tag{8}$$

has the form $p_1(r,\theta) = q(r)\cos\theta$, hence the function $q(r)$ is determined from the equation

$$\int_0^r \left[K\left(\frac{\rho}{r}\right)-E\left(\frac{\rho}{r}\right)\right] q(\rho)\,d\rho + \frac{1}{r}\int_r^a \left[K\left(\frac{r}{\rho}\right)-E\left(\frac{r}{\rho}\right)\right]\rho q(\rho)\,d\rho = \frac{1}{4}b(r)$$

The value of the following integral

$$\int_0^{2\pi} \frac{\cos\varphi\,d\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}} = 2\cos\theta \begin{cases} \dfrac{2}{\rho}\left[K\left(\dfrac{\rho}{r}\right)-E\left(\dfrac{\rho}{r}\right)\right], & \left|\dfrac{\rho}{r}\right|<1 \\ \dfrac{2}{r}\left[K\left(\dfrac{r}{\rho}\right)-E\left(\dfrac{r}{\rho}\right)\right], & \left|\dfrac{r}{\rho}\right|<1 \end{cases}$$

was used in deriving this equation, where $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kinds.

The expression for the function $q(r)$ can be obtained directly also by using (6) or the method for solving (8) proposed in /10/.

The following terms in the series (4) will have the form

$$p_n(r,\theta) = \sum_{k=1}^{n} q_{nk}(r)\cos k\theta$$

as can be seen by evaluating the integral

$$\int_0^{2\pi} \frac{\cos(n\varphi)(r\cos\theta-\rho\cos\varphi)\,d\varphi}{r^2+\rho^2-2r\rho\cos(\theta-\varphi)} = \begin{cases} \dfrac{\pi\rho^n}{r^{n+1}}\cos(n+1)\theta, & \left|\dfrac{\rho}{r}\right|<1 \\ -\dfrac{\pi r^{n-1}}{\rho^n}\cos(n-1)\theta, & \left|\dfrac{\rho}{r}\right|>1 \end{cases}$$

and using the result in /2,10/ asserting that if the shape of the stamp surface being inserted in an elastic half-space is expressed by the formula

$$f(r,\varphi) = f_0(r) + \sum_{m=1}^{n} f_m^{(c)}(r)\cos m\varphi + \sum_{m=1}^{n} f_m^{(s)}(r)\sin m\varphi$$

then the pressure $q(r,\varphi)$ under the stamp will have the form

$$q(r,\varphi) = q_0(r) + \sum_{m=1}^{n} q_m^{(c)}(r)\cos m\varphi + \sum_{m=1}^{n} q_m^{(s)}(r)\sin m\varphi$$

In this case we consider the right side of the integral equation (5) as the function $f(r,\varphi)$ and the form of the solution governs the function $p_n(x,y)$ $(n = 1, 2, \ldots)$.

In the case when the shape and size of the contact area are given in advance, the pressure on the boundary can have a singularity. In this case we seek the solution of the integral equation (3) in the form

$$p(x,y) = \psi(x,y)\chi(x,y)$$

where $\chi(x,y)$ is the extracted singular part, and $\psi(x,y)$ is a regular function that can be determined by the method of expansion in a small parameter described earlier.

We find the singularity of the pressure function $p(x,y)$ as the point (x,y) approaches a certain point (x_0,y_0) belonging to the contour of the loading domain, which is the angular line of the stamp. Evidently only the nearest neighborhood Ω_1 of the point M will exert influence on the singularity of the behavior of the solution of the integral equation (3) at the boundary point $M(x_0,y_0)$. We introduce a coordinate system x', y' at this point by directing the x' axis along the normal and the y' axis along the tangent to the boundary of the contact area Ω at the point M. The axes x', y' will make the angle θ with the axes x, y. We decompose the stamp motion at the point M into two components, with velocity $v\cos\theta$ along the x' axis and with velocity $-v\sin\theta$ along the y' axis. Correspondingly, we decompose the state of stress in the domain Ω_1, which we select symmetric with respect to y' $(-\alpha \leqslant x' \leqslant 0,\ -\beta \leqslant y' \leqslant \beta)$, into the components $\tau_{x'z} = \tau_{xz}\cos\theta$ and $\tau_{y'z} = -\tau_{xz}\sin\theta$.

We consider the integral equation (3) at the boundary point (x_0, y_0) of the contact area and we separate the integral in the left side of (3) into two: in the domain Ω_1 in the (x', y') coordinate system with origin at the point (x_0, y_0), and in the remaining domain $\Omega_2 = \Omega \setminus \Omega_1$. We will then have

$$\iint_{\Omega_1} p(x', y')\left[\frac{1}{\sqrt{x'^2+y'^2}} - \frac{\mu\alpha\cos\theta x'}{x'^2+y'^2} + \frac{\mu\alpha\sin\theta y'}{x'^2+y'^2}\right]dx'\,dy' + \tag{9}$$
$$\iint_{\Omega_2} p(\xi,\eta)\left[\frac{1}{\sqrt{(x_0-\xi)^2+(y_0-\eta)^2}} + \frac{\mu\alpha(x_0-\xi)}{(x_0-\xi)^2+(y_0-\eta)^2}\right]d\xi\,d\eta =$$
$$\frac{\pi E}{1-\nu^2}[g(x_0,y_0)+\delta]$$

As in all contact problems, the singularity of the function $p(x',y')$ on the boundary is of power-law nature, i.e., $p(x',y') = \psi(x',y')/(\sqrt{x'^2+y'^2})^{\beta}$ and is determined by the first integral component on the left side of (9). Because of the symmetry of the domain Ω_1 $(\Omega_1 \to 0)$ with respect to the x' axis and the regularity of the function $\psi(x',y')$ in the domain Ω_1, the integral

$$\iint_{\Omega_1} \psi(x',y')\frac{y'}{(x'^2+y'^2)^{1+\beta/2}}\,dx'\,dy'$$

due to the action of forces directed along the tangent to the boundary, will not, compared to the integral

$$\iint_{\Omega_1} \frac{\psi(x',y')}{(\sqrt{x'^2+y'^2})^{\beta}}\left[\frac{1}{\sqrt{x'^2+y'^2}} - \frac{\mu\alpha\cos\theta x'}{x'^2+y'^2}\right]dx'\,dy'$$

exert influence on the nature of the singularity of the function $p(x',y')$. This deduction agrees with the fact established in /2/ that in the case of friction forces possessing axial symmetry and directed perpendicularly to the radius of the contact area, the equations to determine the stress and strain components decompose into two independent groups and the presence of tangential forces will not affect the magnitudie of the pressure on the contact area. We hence seek the nature of the singularity in the pressure function at the point M by considering the motion of the nearest neighborhood Ω_1 of the point M along the normal to the boundary of the contact domain at this point. In this case it can be considered that the domain Ω_1 is under the conditions of the plane problem.

We therefore have a plane contact problem with friction forces when the friction coefficient equals $\mu_1 = \mu\cos\theta$. We use the solution of this problem, presented in /2/, and we determine the singular part of the pressure function $\chi(x,y)$ in the neighborhood of the point M. For definiteness, we consider a circle of radius a as the domain Ω. In this case the x' axis coincides with the radius of the contact area. Then the pressure on the edges of the contact area will have a singularity of the form (for convenience we introduce a polar coordinate system)

$$\chi(r,\theta) = \frac{1}{(a-r)^{1/2+\gamma}}, \quad \gamma = \frac{1}{\pi}\operatorname{arctg}(\mu_1\alpha) = \frac{1}{\pi}\operatorname{arctg}(\varepsilon\cos\theta)$$

Therefore, the pressure distribution on the contact area under a cylindrical stamp with angular line on the boundary of the contact domain should be sought in the form

$$p(r,\theta) = \frac{\psi(r,\theta)}{(a-r)^{1/2+\gamma}}$$

The function $\psi(r,\theta)$ is bounded everywhere and continuous. An equation follows from (3) and (5), which the function $\psi(r,\theta)$ should satisfy

$$\int_0^{2\pi}\int_0^a \frac{\psi(\rho,\varphi)\rho}{(a-\rho)^{1/2+\gamma}}\left[\frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}} + \frac{\varepsilon(r\cos\theta-\rho\cos\varphi)}{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}\right]d\rho\,d\varphi = \frac{\pi E}{1-\nu^2}[g(r)+\delta]$$

where $g(r)$ is the shape of the lateral surface of the stamp which is a body of revolution. Furthermore, to find the function $\psi(r,\theta)$ we can apply the method of expansion in a series in a small parameter, elucidated above, and can construct a recurrent system of equations to determine the desired terms of the series $\psi_k(r,\theta)$. The mentioned method of extracting the singularity can be applied only for sufficiently smooth shapes of the contact contours. In the case of a contact area with angular points, the general method of extracting singularities, elucidated in /11/, should be utilized.

Finally, we consider the case of motion of a stamp of circular planform along the boundary of a rough elastic half-space. We assume that additional normal displacements of the boundary of the elastic base are proportional to the pressure $p(r,\theta)$ because of deformation $w_*(r,\theta)$ of the microprojections

$$w_*(r,\theta) = \varkappa p(r,\theta) \tag{10}$$

where $\varkappa$ is a coefficient characterizing the deformation properties of the rough layer.

The normal displacements of the elastic half-space boundary are comprised of elastic displacements of points of the half-space boundary, determined by (3), and additional displacements because of deformation $w_*(r,\theta)$ of the microprojections (10). We write the contact condition between the stamp and the half-space boundary

$$\varkappa p(r,\theta) + \int_0^a \int_0^{2\pi} p(\rho,\varphi)\left[\frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}} + \varepsilon\frac{r\cos\theta-\rho\cos\varphi}{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}\right]\rho\,d\rho\,d\varphi = g(r)+\delta \tag{11}$$

We note that the solution of (11) cannot become infinite at the ends of the contact area.

In fact, assuming the pressure to have an integrable power-law singularity of the form $(a-r)^{-\beta}$ $(0<\beta<1$, and a is a point on the boundary and taking into account that the integral term in (11) has no singularity on the domain boundary, we obtain that the left side of (11) has a singularity of the order of $(a-r)^{-\beta}$, while there is no singularity on the right side. The contradiction obtained indeed proves the assertion expressed above.

As before, we represent the pressure function $p(r,\theta)$,which is bounded everywhere, in the form of the series

$$p(r,\theta) = p_0(r) + \varepsilon p_1(r,\theta) + \ldots + \varepsilon^n p_n(r,\theta) + \ldots \tag{12}$$

and substitute it into (11). We then obtain a recurrent system of equations to determine the unknown functions $p_n(r,\theta)$

$$\begin{aligned} &A[p_0(r)] + \varkappa p_0(r) = g(r)+\delta \\ &A[p_n(r,\theta)] + \varkappa p_n(r,\theta) = B[p_{n-1}(r,\theta)], \quad n=1,2,\ldots \end{aligned} \tag{13}$$

Therefore, to determine the required functions $p_n(r,\theta)$ an inhomogeneous Fredholm integral equation of the second kind (13) must be solved in each step. Its solution can be obtained by successive approximations /12/.

Convergence of the series (12) is proved analogously to the preceding since the operator A^* in the left sides of (13) are bounded as the sum of two bounded operators.

In all the cases considered, the pressure on the contact area is represented by the series (4), which indicates particularly that the pressure under a stamp of circular planform will be distributed nonsymmetrically during its motion, whereupon an additional moment M_y with respect to the oy axis will occur

$$M_y = \int_0^a\int_0^{2\pi} p(\rho,\varphi)\rho^2\cos\varphi\,d\rho\,d\varphi = \varepsilon\int_0^a\int_0^{2\pi} p_1(\rho,\varphi)\rho^2\cos\varphi\,d\rho\,d\varphi + O(\varepsilon^2)$$

The equivalent pressure P will be displaced in the direction of stamp motion at a distance d from its axis, which can be determined to second order accuracy from the formula

$$d = \frac{M_y}{P} = \frac{\varepsilon\pi}{P}\int_0^a\int_0^{2\pi} p_1(\rho,\varphi)\rho^2\cos\varphi\,d\rho\,d\varphi$$

As was shown earlier, the function for the pressure has the form $p(r,\theta) = p_0(r) + \varepsilon q(r)\cos\theta + O(\varepsilon^2)$ during displacement of a smooth stamp of circular planform along an elastic half-space boundary. In this case the moment M_y and the displacement of the equivalent d are determined from the formulas

$$M_y = \varepsilon\pi\int_0^a q(\rho)\rho^2\,d\rho + O(\varepsilon^2), \quad d = \frac{\varepsilon\pi}{P}\int_0^a q(\rho)\rho^2\,d\rho$$

From the equilibrium condition for the moments of forces acting on a moving stamp it follows that the force T causing stamp motion and directed along the ox axis should be applied at a distance h from the base, where

$$h = \frac{M_y}{T} = \frac{M_y}{\mu P} = \frac{\varepsilon\pi}{\mu P} \int_0^a q(\rho)\rho^2 dp = \frac{\alpha\pi}{P} \int_0^a q(\rho)\rho^2 d\rho \tag{14}$$

When (14) is not satisfied, the stamp will have an oblique base, which implies a change in the boundary conditions (1). In the case of a flat stamp of circular planform the function $w(r, \theta)$ in (1) will have the form

$$w(r, \theta) = \beta r \cos\theta + \delta$$

The unknown constant β governing the slope of the stamp can be found from the condition of equality of the moments of all forces acting on the stamp.

REFERENCES

1. VANTORIN V.D., Motion along a plane with anisotropic friction. In: Friction and Wear in Machines, Vol.16, Izdat. Akad. Nauk SSSR, Moscow, 1962.
2. GALIN L.A., Contact Problems of Elasticity and Viscoelasticity Theories.NAUKA, Moscow, 1980.
3. ABRAMIAN B.L., ARUTIUNIAN N.Kh. and BABLOIAN A.A., On symmetric pressure of a circular stamp on an elastic half-space in the presence of adhesion, PMM, Vol.30, No.1, 1966.
4. MOSSAKOVSKII V.I., The fundamental mixed problem of the theory of elasticity for a half-space with a circular line of separation of the boundary conditions. PMM, Vol.18, No.2, 1954.
5. LUR'E A.I., Three-dimensional Problem of Elasticity Theory. GOSTEKHIZDAT, Moscow, 1955.
6. KRAVCHUK A.S., Solution of certain three-dimensional contact problems with friction taken into account on the contiguous surfaces. Trenie i Iznos, Vol.2, No.4, 1981.
7. TRICOMI F., Integral Equations, N.Y. Interscience, 1957.
8. ZABREIKO P.P., KOSHELEV A.I., KRASNOSEL'SKII M.A., MIKHLIN S.G., RAKOVSHCHIK L.S. and STETSENKO V.Ia., Integral Equations. NAUKA, Moscow, 1968.
9. KOLMOGOROV A.N. and FOMIN S.V., Elements of the Theory of Functions and Functional Analysis. NAUKA, Moscow, 1968.
10. ROSTOVTSEV N.A., Complex potentials in the problem of a punch of circular section, PMM, Vol.21, No.1, 1957.
11. BABESHKO V.A., GLUSHKOV E.V. and GLUSHKOVA N.V., On singularities at angular points of three-dimensional stamps in contact problems. Dokl. Akad. Nauk SSSR, Vol.257, No.2, 1981.
12. GORIACHEVA I.G., Plane and axisymmetric contact problems for rough elastic bodies, PMM, Vol.43, No.1, 1979.

Translated by M.D.F.